11.

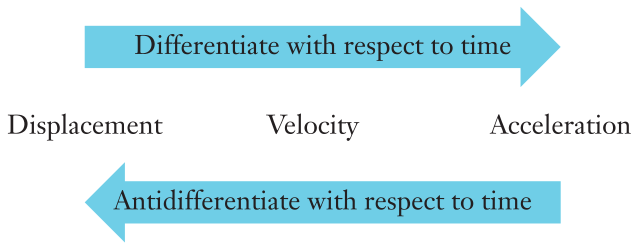
Simple harmonic motion

- Revision of rectilinear motion
- A particular type of rectilinear motion: Simple harmonic motion
- Solving the differential equation $\ddot{x} = -k^2 x$
- Miscellaneous exercise eleven

Revision of rectilinear motion

Before investigating simple harmonic motion, the title of this chapter, let us first refresh our knowledge of general motion in a straight line, and also extend that knowledge a little.

From your previous study of Units Two and Three of *Mathematics Methods* you should be familiar with the following:



EXAMPLE 1

A particle is initially at an origin O. It is projected away from O and moves in a straight line such that its displacement from O, t seconds later, is x metres where $x = t(10 - t)$.

Find **a** the speed the particle when $t = 12$,

b the value of t when the particle comes to rest and the distance from the origin at that time,

c the distance the particle travels from $t = 3$ to $t = 6$.

Solution

a If $x = 10t - t^2$ then $v = 10 - 2t$

Thus when $t = 12$ $v = -14$ m/s

The speed when $t = 12$ is 14 m/s.

b With $v = 10 - 2t$ then the particle being at rest means $10 - 2t = 0$, i.e. $t = 5$.

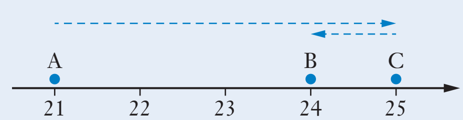
When $t = 5$, $x = 5(10 - 5)$ i.e. $x = 25$.

The particle is at rest when $t = 5$ and it is then 25 m from O.

c When $t = 3$ the particle is 21 m from O and when $t = 6$ it is 24 m from O.

However the distance travelled in this time is not simply $(24 - 21)$ m. Our answer to **b** indicates that the particle stopped when $t = 5$, at $x = 25$ m.

From $t = 3$ to $t = 6$ the particle travels from A to C to B (see diagram).



The particle travels 5 m from $t = 3$ to $t = 6$.

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EXAMPLE 2

A particle travels along a straight line with its velocity at time t seconds given by v m/s where $v = 6t^2 - 3$.
The initial displacement of the particle from a point O on the line is 5 metres.
Find the displacement of the particle from O when $t = 5$.

Solution

$$x = \int v\,dt = \int (6t^2 - 3)\,dt$$
$$= 2t^3 - 3t + c$$

We know that initially, i.e. when $t = 0$, $x = 5$.

$\therefore \quad 5 = 2(0)^3 - 3(0) + c \qquad$ i.e. $c = 5$.

Thus $\quad x = 2t^3 - 3t + 5 \qquad \therefore$ When $t = 5$, $x = 240$.

When $t = 5$ the displacement from O is 240 metres.

The previous two examples involved functions given in terms of t, the time. This may not always be the case. We could, for example, be given the velocity, v, in terms of the displacement, x, (as we saw in chapter 8). The next examples involve such situations, but first note the following useful rearrangement.

$$\text{Acceleration} = \frac{dv}{dt}$$
$$= \frac{dv}{dx}\frac{dx}{dt}. \qquad \text{But } \frac{dx}{dt} = v.$$

Hence, $\quad$ acceleration $= v\dfrac{dv}{dx}$. $\qquad$ This could also be written as $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$.

In the examples that follow assume that x m, v m/s and a m/s^2 represent displacement, velocity and acceleration respectively.

EXAMPLE 3

If $v = 3x$ find the acceleration when $x = 2$.

Solution

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$
$$= v\frac{dv}{dx}$$
$$= (3x)(3)$$
$$= 9x$$

Thus when $x = 2$, $a = 18$. When $x = 2$ the acceleration is 18 m/s^2.

Repeat the previous example but instead write $\dfrac{dx}{dt} = 3x$, integrate using separation of variables to find an expression for x and then differentiate twice to find the acceleration.

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EXAMPLE 4

$a = 2x + 4$, and when $x = 1$, $v = 2$. Find v, $v \geq 0$, when $x = 3$.

Solution

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$
$$= v\frac{dv}{dx}$$

Thus $$v\frac{dv}{dx} = 2x + 4$$

Separating the variables and integrating: $$\int v\,dv = \int (2x + 4)\,dx$$
$$\frac{v^2}{2} = x^2 + 4x + c$$

When $x = 1$, $v = 2$, thus $$c = -3$$

Therefore $$\frac{v^2}{2} = x^2 + 4x - 3$$

When $x = 3$ $$v = 6 \qquad (v \geq 0).$$

EXAMPLE 5

$a = \frac{2}{v}$, $v > 0$, and initially, i.e. when $t = 0$, $x = 8$ and $v = 3$.

a Find v, $v > 0$, when $t = 10$.

b Find v, $v > 0$, when $x = 10$.

Solution

a This part involves t and v.

Write $\frac{dv}{dt} = \frac{2}{v}$

$$\int v\,dv = \int 2\,dt$$
$$\frac{v^2}{2} = 2t + c$$

When $t = 0$, $v = 3$ and so $c = 4.5$.

$$v^2 = 4t + 9$$

Thus when $t = 10$, $v = 7$.

b This part involves x and v.

Write $v\frac{dv}{dx} = \frac{2}{v}$

$$\int v^2\,dv = \int 2\,dx$$
$$\frac{v^3}{3} = 2x + d$$

When $x = 8$, $v = 3$ and so $d = -7$.

$$v^3 = 6x - 21$$

Thus when $x = 10$, $v = \sqrt[3]{39}$.

Exercise 11A

Questions **1** to **8** all involve rectilinear motion with x metres, v m/s and a m/s^2 the displacement, velocity and acceleration of a body respectively, relative to an origin O, at time t seconds.

1 If $v = 6t\sqrt{16 + t^2}$ find **a** the acceleration when $t = 0$,
b the displacement when $t = 3$ if when $t = 0$, $x = 8$.

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2 $a = \frac{6t(t+1)^2}{5}$. Find the velocity when $t = 0$ given that when $t = 1$, $v = 2$ m/s.

3 If $x = 5 + 2\cos t$ find **a** the velocity when $t = \frac{\pi}{6}$, **b** the acceleration when $t = \frac{\pi}{2}$.

4 If $v = 4\sin 2t$ find **a** the acceleration when $t = \frac{\pi}{6}$,

b the displacement when $t = \frac{\pi}{2}$ given that for $t = 0$, $x = 3$.

5 If $a = 4\sin t\cos t$ and when $t = 0$, the displacement is 5 m and the velocity is 3 m/s, find

a the velocity when $t = \frac{\pi}{3}$, **b** the displacement when $t = \frac{\pi}{3}$.

6 If $v = 5 + x^2$ find the acceleration when $x = 1$.

7 If $a = 3x^2 + 1$ and when $x = 0$, $v = 2$, find the velocity, $v > 0$, when $x = 3$.

8 If $a = v^2$, $v > 0$, and when $t = 2$, $x = 0$ and $v = 0.1$, find

a v when $t = 10$, **b** v when $x = 2$.

9 The displacement of a body from an origin O, at time t seconds, is x metres, where

$$x = \frac{t+1}{2t+3}, \qquad t \geq 0.$$

Find **a** expressions for the velocity and acceleration of the body in terms of t,

b the displacement, velocity and acceleration of the body when $t = 1$.

10 An object projected vertically upwards from a point above ground level is h metres above ground level t seconds later, where $h = 42 + 29t - 5t^2$ $(t \geq 0)$. For what value of t, and at what speed, does the object hit the ground?

11 A particle is initially at an origin O. It is projected away from O and moves in a straight line such that its displacement from O, t seconds later, is x metres, where

$$x = t(16 - t).$$

Find **a** the speed of the particle when $t = 20$,

b the value of t when the particle comes to rest and the distance from the origin at that time,

c the distance the particle travels from $t = 1$ to $t = 5$,

d the distance the particle travels from $t = 5$ to $t = 10$.

12 A body is initially at an origin, O. At that instant the velocity of the body is 35 m/s. The acceleration, t seconds later, is $6(t - 4)$ m/s^2. Find the velocity of the body when it is **next** at O.

13 A particle passes through an origin O at time $t = 0$ and travels along a straight line such that its velocity t seconds later, is v m/s where

$$v = 2\sin 2t$$

Determine **a** the maximum velocity of the particle during the motion,

b an expression for the acceleration of the particle at time t,

c the maximum acceleration of the particle during the motion,

d an expression for the displacement of the particle at time t,

e the maximum displacement of the particle during the motion.

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14 A particle moves in a straight line such that its velocity, v m/s, depends upon displacement, x m, from some fixed point O according to the rule

$$v = 3x + 2$$

Determine **a** an expression in terms of x for the acceleration of the particle,

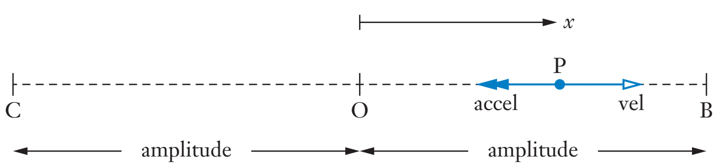
b the velocity and acceleration of the particle when $x = 4$.

15 A particle is initially (i.e. $t = 0$) at an origin (i.e. $x = 0$) and moving with a velocity of 4 m/s. The particle moves such that its acceleration at any time is a function of its velocity at that time, with acceleration $= -(1 + v^2)$ m/s^2.

Find the exact distance the particle moves in reducing its velocity to one-quarter of its initial velocity.

A particular type of rectilinear motion: Simple harmonic motion

Suppose that an object moves along a straight line with its acceleration proportional to its displacement from some fixed point, O, on the line, and always directed towards O. As the object gets further away from O it will experience an increasing 'pull' back towards O. This will cause the object to **oscillate** about O as shown below:



In the diagram the fixed point, or **mean position**, is O. The object is shown at some point P, displacement x from O, and is travelling away from O with velocity v. The acceleration is towards O and will cause the object to slow down. If it *just* reaches point B we call OB the **amplitude** of the motion. The acceleration will then cause the object to travel back through O and to *just* reach C, then returning through O to just reach B again, and so on.

This oscillatory motion is called **simple harmonic motion**.

With the acceleration proportional to the displacement then

$$\frac{d^2x}{dt^2} \text{ is proportional to } x.$$

In mathematics we write this: $\ddot{x} \propto x$

Introducing k^2 as the constant of proportionality we have

$$\ddot{x} = -k^2x$$

A squared constant is used to make later integration more straightforward and the negative sign is because the acceleration is always directed towards O.

> If a body moves such that $\ddot{x} = -k^2x$
>
> then the body is moving with **simple harmonic motion (SHM)**.

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Solving the differential equation $\ddot{x} = -k^2x$

As we have already seen in the previous chapter:

- Any equation that involves one or more derivatives, e.g. $\frac{dy}{dx}$, $\frac{dp}{dt}$, $\frac{d^2x}{dt^2}$, etc, is called a **differential equation**.
- To *solve* a differential equation we must find a relationship between the variables involved, that satisfies the differential equation, but that does not contain any derivatives.

Thus, to solve $\ddot{x} = -k^2x$ we need to find a relationship between x and t, not involving derivatives.

From $\quad \boldsymbol{\ddot{x} = -k^2x}$

it follows that $\quad \frac{dv}{dt} = -k^2x$.

Hence, by the chain rule $\quad \frac{dv}{dx}\frac{dx}{dt} = -k^2x.$

But $\frac{dx}{dt} = v$ and so $\quad v\frac{dv}{dx} = -k^2x.$

Separating the variables $\quad \int v\,dv = -\int k^2x\,dx$

$$\frac{v^2}{2} = -\frac{k^2x^2}{2} + c$$

But if 'a' is the amplitude then when $x = a$, $v = 0$.

Hence $\quad 0 = -\frac{k^2a^2}{2} + c$

i.e. $\quad c = \frac{k^2a^2}{2}$

Thus $\quad \boldsymbol{v^2 = k^2(a^2 - x^2)}$

where a is the amplitude of the motion.

Taking the positive square root $\quad v = k\sqrt{a^2 - x^2}$

i.e. $\quad \frac{dx}{dt} = k\sqrt{a^2 - x^2}$

Separating the variables $\quad \int \frac{dx}{\sqrt{a^2 - x^2}}dx = \int k\,dt$

Using the substitution $\quad x = a\sin u$, from which $\frac{dx}{du} = a\cos u$,

$$\int 1\,du = \int k\,dt$$

i.e. $\quad u = kt + \alpha \qquad$ (α is the integration const.)

$\therefore \quad \boldsymbol{x = a\sin(kt + \alpha)}$

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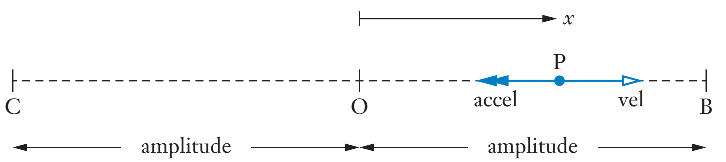
Hence, as we might have expected, the motion is periodic. Wherever the object is at some time t_1 it will be there again at time $t_1 + \frac{2\pi}{k}$. **The time period is** $\boldsymbol{\frac{2\pi}{k}}$.

The constant α (the Greek letter, 'alpha') is the **phase angle** and depends on the position of the object when timing commences, i.e. when $t = 0$.



The following points follow from the displacement equation $\quad x = a\sin(kt + \alpha)$.

- Suppose timing commences when the object is at O (see diagram above for O).

  Then $x = 0$ when $t = 0$. Thus $\quad 0 = a\sin\alpha \quad$ allowing us to have $\alpha = 0$.

  In such cases $\quad x = a\sin kt$

- Suppose timing commences when the object is at B (see diagram above for B).

  Then $x = a$ when $t = 0$. Thus $\quad a = a\sin\alpha \quad$ allowing us to have $\alpha = \frac{\pi}{2}$.

  In such cases
  $$x = a\sin\left(kt + \frac{\pi}{2}\right)$$
  $$= a\cos kt$$

- The velocity is given by: $\quad v = \dot{x} = ak\cos(kt + \alpha)$

  Thus the extreme values of v are $\pm\, ak$ and they occur when $\cos(kt + \alpha) = \pm 1$. At such points $x\ (= a\sin(kt + \alpha))$ will be zero. Thus the extreme velocities occur as the object passes through O.

  Earlier we found that $\quad v^2 = k^2(a^2 - x^2) \quad$ which also shows that the extreme values of v are $\pm ka$ and occur when $x = 0$.

Summary

- For a body moving with SHM about $x = 0$, $\quad \ddot{x} = -k^2 x$.
- A solution to this equation is $\quad x = a\sin(kt + \alpha)$.
- If timing commences at O (see diagram above) $\quad x = a\sin kt$.
- If timing commences at B (see diagram above) $\quad x = a\cos kt$.
- The motion is periodic with period T, where $\quad T = \frac{2\pi}{k}$.
- The amplitude of the motion is $|a|$.
- The velocity, v, at time t is given by $\quad v = ak\cos(kt + \alpha)$

  It also follows that $\quad v^2 = k^2(a^2 - x^2)$

  and $\quad v_{max} = |ka|$.

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From earlier work involving trigonometric functions we know that many trigonometrical expressions can be written in other ways.

For example we can write $\sin\theta$ as $\cos\left(\theta - \frac{\pi}{2}\right)$,

we can write $\sin(\theta + \alpha)$ as $\sin\theta\cos\alpha + \cos\theta\sin\alpha$ etc.

Hence it may come as no surprise that $x = a\sin(kt + \alpha)$ is not the only way of expressing the solution to $\frac{d^2x}{dt^2} = -k^2x$.

> Show that $x = a\cos(kt + \beta)$
>
> and $x = C\cos kt + D\sin kt$
>
> each satisfy the equation $\frac{d^2x}{dt^2} = -k^2x$.

Note: Earlier, when solving $\ddot{x} = -k^2x$

we obtained the equation $v^2 = k^2(a^2 - x^2)$.

Then 'taking the positive square root', i.e. $v = k\sqrt{a^2 - x^2}$

we wrote this as $\frac{dx}{dt} = k\sqrt{a^2 - x^2}$

separated the variables and integrated using the substitution

$$x = a\sin u.$$

Repeat this process but instead use the negative square root, separate the variables and integrate using the substitution

$$x = a\cos u.$$

EXAMPLE 6

Determine the period of the simple harmonic motion defined by the equation

$$\ddot{x} = -16\,x.$$

Solution

Comparing $\ddot{x} = -16\,x$ with $\ddot{x} = -k^2x$ gives $k = 4$.

Using $T = \frac{2\pi}{k}$ $\quad T = \frac{2\pi}{4}$

$$= \frac{\pi}{2}.$$

The period of the simple harmonic motion is $\frac{\pi}{2}$ seconds.

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EXAMPLE 7

A body moves such that its displacement from an origin O at time t seconds is x metres, where $x = 4\sin 2t$.

a Prove that the motion is simple harmonic.

b Determine the period and amplitude of the motion.

c How far does the body move in the first second?

Solution

a To prove SHM we must show that $\ddot{x} = -k^2x$.

We are given that $x = 4\sin 2t$

Therefore $\dot{x} = 8\cos 2t$

and $\ddot{x} = -16\sin 2t$

$= -4x$

This is of the form $\ddot{x} = -k^2x$.

The motion is simple harmonic.

b From **a**, or by comparing $x = 4\sin 2t$ with $x = a\sin kt$, we see that $k = 2$.

The time period, T, is given by $T = \dfrac{2\pi}{k}$

$= \pi$

The time period is π seconds.

Comparing $x = 4\sin 2t$ with $x = a\sin kt$ gives $a = 4$.

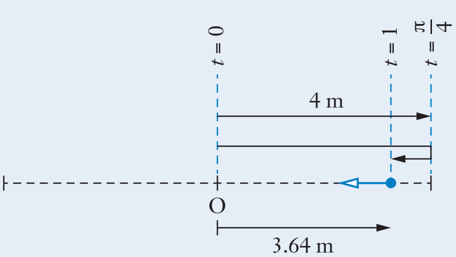
The amplitude of the motion is 4 metres.

c When $t = 0$ $x = 0$.

When $t = 1$ $x = 4\sin 2$

$= 3.64$ (to 2 decimal places).

But when $t = \dfrac{\pi}{4}$ $x = 4$.



Thus the distance travelled in the first second is $4 + (4 - 3.64) = 4.36$ m.

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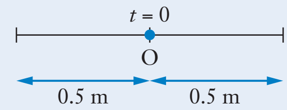
EXAMPLE 8

A body moves with SHM about some mean position O. The amplitude of the motion is 0.5 metres, the period is 2π seconds and when $t = 0$ the body is at O.

Find **a** the two possible expressions of the form $x = a\sin(kt)$ for the displacement of the body from O at time t,

b the speed of the body when $t = \dfrac{\pi}{3}$.



Solution

a The amplitude is 0.5 m and when $t = 0$ the body is at O.

Therefore $x = \pm 0.5\sin kt$

(The ± is necessary because at $t = 0$ motion may be to the right or to the left. We are not told whether the displacement is positive or negative for small positive t.)

The period is 2π, therefore $\dfrac{2\pi}{k} = \pi$ giving $k = 1$.

The displacement of the body from O at time t is x m where $x = \pm 0.5\sin t$.

b If $x = \pm 0.5\sin t$ then $\dot{x} = \pm 0.5\cos t$

When $t = \dfrac{\pi}{3}$ $\dot{x} = \pm 0.25$

When $t = \dfrac{\pi}{3}$ the body has speed 0.25 m/s.

EXAMPLE 9

A body moves with SHM about some mean position O. The amplitude of the motion is 2 metres, the period is $\dfrac{\pi}{5}$ seconds and when $t = 0$ the displacement of the body from O is 1 metre and the velocity is positive.

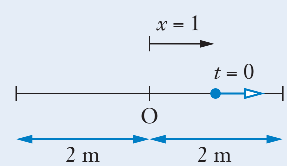
Find **a** an expression for the displacement of the body from O at time t, giving your answer in the form $x = a\sin(kt + \alpha)$ for $0 \le \alpha \le \pi$,

b the greatest speed attained by the body,

c the greatest acceleration of the body.

Solution

a The amplitude is 2 m therefore $x = 2\sin(kt + \alpha)$.



The period is $\dfrac{\pi}{5}$, therefore $\dfrac{2\pi}{k} = \dfrac{\pi}{5}$ i.e. $k = 10$

Thus $x = 2\sin(10t + \alpha)$

When $t = 0$, $x = 1$, therefore $1 = 2\sin\alpha$

giving $\alpha = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

If $x = 2\sin\left(10t + \dfrac{\pi}{6}\right)$ $v = 20\cos\left(10t + \dfrac{\pi}{6}\right)$

and if $x = 2\sin\left(10t + \dfrac{5\pi}{6}\right)$ $v = 20\cos\left(10t + \dfrac{5\pi}{6}\right)$

Only the first of these velocities is positive when $t = 0$.

The displacement from O at time t is x m where $x = 2\sin\left(10t + \dfrac{\pi}{6}\right)$

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b If $x = 2\sin\left(10t + \frac{\pi}{6}\right)$ then $\dot{x} = 20\cos\left(10t + \frac{\pi}{6}\right)$

and so $\dot{x}_{max} = 20$

The maximum speed is 20 m/s.

(Alternatively, obtain this same answer using $v_{max} = |ka|$.)

c Now $\ddot{x} = -k^2x$ so, in this case, $\ddot{x} = -100x$

and so $\ddot{x}_{max} = -100(-2)$

$= 200$

The greatest acceleration of the body is 200 m/s^2.

Exercise 11B

1 With x metres representing the displacement of a body at time t seconds, determine the amplitude and period of SHM with

a $x = 5\sin 2t$, **b** $x = 4\sin 5t$, **c** $x = 2\cos 4t$.

2 With x metres representing the displacement of a body at time t seconds, determine the period of the simple harmonic motion defined by the equation

a $\ddot{x} = -4x$, **b** $\ddot{x} = -x$, **c** $\ddot{x} = -25x$.

3 A particle moves with simple harmonic motion about some mean position O and when timing commences the particle is at O. Write an equation for x, the displacement of the particle from O in metres, t seconds later, given that the motion has

a amplitude 1 m, period 4π seconds and an initial positive velocity,

b amplitude 1 m, period 4π seconds and an initial negative velocity,

c amplitude 3 m and period π seconds and an initial positive velocity,

d amplitude 0.5 m and period 2 seconds and an initial negative velocity.

4 A particle moves with simple harmonic motion about some mean position O and when timing commences the particle is at its maximum displacement from O. Write an equation for x, the displacement of the particle from O in metres, t seconds later given that the motion has

a amplitude 2 m and period π seconds,

b amplitude 1.5 m and period 0.5π seconds,

c amplitude 0.5 m and period 0.5 seconds.

5 A body moves with simple harmonic motion about some mean position O. The amplitude of the motion is 2.5 metres, the period is π seconds and initially (i.e. when $t = 0$) the body is at O.

Find **a** the two possible expressions of the form $x = a\sin kt$ for the displacement of the body from O at time t,

b the speed of the body when $t = \frac{\pi}{6}$.

6 With x metres representing the displacement of a body at time t seconds, determine the amplitude and period of SHM with:

a $x = 5\cos 5t + 3\sin 5t$ **b** $x = 3\cos 2t + 7\sin 2t$

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7 A body moves such that its displacement from an origin O at time t seconds is x m, where $x = 4\sin\frac{\pi t}{10}$.

a Prove that the motion is simple harmonic.

b Determine the period and amplitude of the motion.

c How far does the body move in the first two seconds (to the nearest cm)?

8 A body moves such that its displacement from an origin O at time t seconds is x m, where $x = 2\sin\frac{\pi t}{3}$.

a Prove that the motion is simple harmonic.

b Determine the period and amplitude of the motion.

c Exactly how far does the body move in the first two seconds?

9 A body moves such that its displacement from an origin O at time t seconds is x m, where $x = 3\sin\left(2t + \frac{\pi}{6}\right)$.

a Prove that the motion is simple harmonic.

b Determine the period and amplitude of the motion.

c How far does the body move in the first second (to the nearest cm)?

10 A body moves with simple harmonic motion about some mean position O. The amplitude of the motion is 4 metres, the period is 2 seconds and initially (i.e. when $t = 0$) the displacement of the body from O is 2 metres and the velocity is negative.

Find **a** an expression for the displacement of the body from O at time t seconds giving your answer in the form $x = a\sin(kt + \alpha)$ for $0 \le \alpha \le \pi$.

b the speed of the body when $t = \frac{1}{6}$.

11 A body moves with simple harmonic motion about some mean position O.

The amplitude of the motion is 2 metres, the period $\frac{2\pi}{5}$ seconds and when $t = 0$ (i.e. initially) the displacement of the body from O is $\sqrt{2}$ m and the velocity is positive.

Find **a** an expression for the displacement of the body from O at time t sec giving your answer in the form $x = a\sin(kt + \alpha)$ for $0 \le \alpha \le \pi$,

b the greatest speed attained by the body,

c the greatest acceleration of the body.

12 A body moves with SHM with equation $\ddot{x} = -4x$, where x m is the displacement of the body from a fixed point O at time t seconds. Initially, i.e. when $t = 0$, the body is at O and has a positive velocity. If the amplitude of the motion is 0.6 m determine

a the displacement of the body from O when $t = \frac{\pi}{6}$,

b the displacement of the body from O when $t = \frac{\pi}{3}$,

c the value of t ($t \ge 0$) when the body is a distance of 0.3 m from O for

i the first time, **ii** the second time, **iii** the third time.

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13 A body moves with SHM with equation $\ddot{x} = -\pi^2 x$, where x m is the displacement of the body from a fixed point O at time t seconds. Initially, i.e. when $t = 0$, the body is at O and has a negative velocity.

If the amplitude of the motion is 3 m determine

a the displacement of the body from O when $t = \frac{1}{3}$,

b the velocity of the body when $t = \frac{1}{3}$,

c the speed of the body when $t = \frac{1}{3}$,

d the value of t ($t \geq 0$) when the body next has the speed it had at $t = \frac{1}{3}$.

14 The five points A, B, C, D and E lie in that order on a straight line such that

$$AB = BE = 3\text{m} \quad \text{and} \quad BC = CD = DE.$$

A particle performs SHM along this line, with time period π seconds, point B as the mean position and points A and E as the extreme positions.

How long, to the nearest 0.01 second, does it take for the particle to travel

a from A to C? **b** from C to D? **c** from D to E?

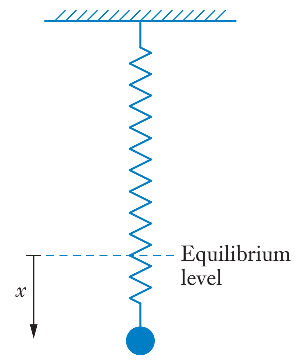
d Determine the two possible times that could elapse from the particle leaving D until it next returns to D (again to nearest 0.01 second).

15 A body moves with simple harmonic motion such that its displacement from the mean position O at time t seconds is x m, where $x = 2\sin 4t$. For how long in each cycle of the motion is the particle at least 1.5 m from O? (Answer to the nearest hundredth of a second.)

16 A body moves with SHM with equation $\ddot{x} = -4x$, where x m is the displacement of the body from a fixed point O at time t seconds. If v m/s is the velocity of the body at time t find an expression for x in terms of t given that

a when $t = 0$, $x = 0$ and $v = 4$, **b** when $t = 0$, $x = 4$ and $v = 0$.

17 The diagram on the right shows a mass hanging from a spring. The mass is pulled down 2 cm below its equilibrium level and released from rest. The mass performs SHM about the equilibrium level such that if x metres is the displacement of the mass from the equilibrium level t seconds after release, then $\ddot{x} = -64x$.



Find **a** the amplitude of the motion,

b the period of the motion,

c the time from release to the mass reaching the equilibrium position for the first time,

d the speed of the mass as it passes through the equilibrium level,

e the time from release to the mass first reaching a speed equal to half of the maximum speed it attains during the motion.

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18 An object moves such that its displacement, x metres, from some fixed point O, at time t seconds, is given by $x = -4\sqrt{3}\sin 2t - 4\cos 2t$.

a How far is the object from O when $t = 0$?

b Prove that for this motion $\ddot{x} = -k^2 x$ and determine the value of k.

c How far does the object move in the first 1.5 seconds (to the nearest cm)?

19 A particle moves such that its displacement, x m, from some fixed point O at time t seconds is given by

$$x = 3 + 4\sin \pi t$$

a Show that this satisfies $\ddot{p} = -k^2 p$ where $p = x - 3$ and k is a constant.

b By completing part **a** you have shown that the motion is simple harmonic. What is the period and amplitude of the motion?

c If point B is the mean position of the SHM, how far is B from O?

d What is the greatest distance that the particle is from O during the SHM?

20 A particle moves such that its displacement, x m, from some fixed point O at time t seconds is given by

$$x = 5 - 3\cos 2t$$

a Show that this satisfies $\ddot{s} = -k^2 s$ where $s = x - 5$ and k is a constant.

b By completing part **a** you have shown that the motion is simple harmonic. What is the period and amplitude of the motion?

c If point P is the mean position of the SHM, how far is P from O?

d What is the least distance that the particle is from O during the SHM?

21 An object moves along the x-axis with its velocity, v m/s, at time t seconds given by:

$$v = \frac{1}{4}\cos t.$$

Determine **a** the distance travelled by the object from $t = 0$ to $t = 1$, giving your answer to the nearest centimetre.

b the distance travelled by the object from $t = 0$ to $t = 2$, giving your answer to the nearest centimetre.

22 A particle moves with SHM about some mean position O with its displacement from O at time t seconds being x m and its velocity at that instant being v m/s.

If $x = 20$ when $v = 30$ and $x = 24$ when $v = 14$ find the period and amplitude of the motion.

23 A particle moves with SHM about some mean position O with its displacement from O at time t seconds being x m and its velocity at that instant being v m/s.

If $x = 0.6$ when $v = 0.75$ and $x = 0.39$ when $v = 1.56$ find the period and amplitude of the motion.

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Miscellaneous exercise eleven

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters in this unit, and the ideas mentioned in the Preliminary work section at the beginning of this unit.

1 Determine $\frac{dy}{dx}$ for each of the following:

a $\ln y = 3x^2$

b $4xy + y^5 - 15x = 4\sin 2x$

2 Clearly showing your use of calculus, find the half-life of a radioactive element that decays according to the rule $\frac{dA}{dt} = -0.02A$, where A is the amount present after t years.

3 A particle passes through an origin O at time $t = 0$ and travels along a straight line such that its velocity t seconds later is v m/s, where

$$v = 3\sin^2 t$$

Determine **a** the minimum velocity of the particle during the motion,

b an expression for the acceleration of the particle at time t,

c the smallest value of t ($t \geq 0$) for which the acceleration is at its maximum value,

d an expression for the displacement of the particle at time t,

e the displacement of the particle when $t = \frac{\pi}{6}$.

4 A particle moves in a straight line such that its velocity, v m/s, depends upon displacement, x m, from some fixed point O according to the rule

$$v = 3x^2 - 2$$

Determine **a** an expression in terms of x for the acceleration of the particle,

b the velocity and acceleration of the particle when $x = 1$.

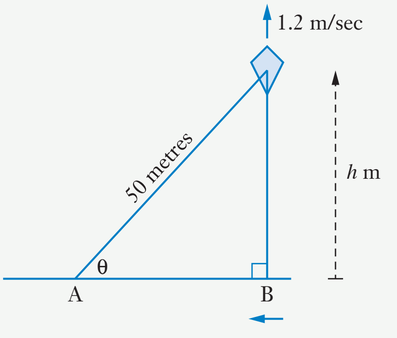
5 Find the equation of a curve having a gradient at the general point (x, y) on the curve equal to $\frac{2x}{3y^2}$ and passing through the point (2, 1).

6 A kite is being controlled from point A on horizontal ground using 50 metres of string.

The string makes a straight line from A to the kite.

The kite is rising vertically at 1.2 m/s and the fixed length of string means that point B on the ground, directly below the kite, approaches A.

If θ radians is the angle of elevation of the kite from point A at time t seconds find $\frac{d\theta}{dt}$ and the rate at which B approaches A when the kite is 40 metres above the ground.



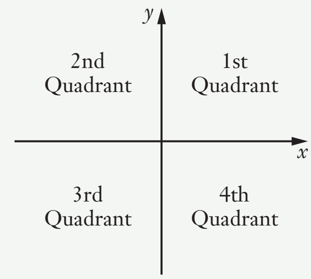
7 Find the volume of the solid formed by rotating the area enclosed between the curve $y = 4\sqrt{x}$ and the x-axis from $x = 3$ to $x = 5$ through one revolution about the x-axis.

8 Find the area bounded by $y = \cos^3 x$ and the x-axis from $x = 0$ to $x = \frac{\pi}{2}$.

9 **a** Find, exactly, the area of the region lying in the first quadrant and enclosed by:

$$\text{the curve } y = 2(x^2 + 1),$$

the x-axis, the y-axis and the line $x = 2$.



b Find, exactly, the volume of the solid of revolution formed by rotating the region described in part **a** through 360° about the x-axis.

c Find, exactly, the volume of the solid of revolution formed by rotating the region described in part **a** through 360° about the y-axis.

10 A particle travels along a straight line with its acceleration at time t seconds equal to $(6t + 4)$ m/s^2. The particle has an initial positive velocity and travels 32 m in the third second. Find the velocity of the body when $t = 1$.

11 In each of the following x m is the displacement of an object from some fixed origin O at time t seconds. Prove that each object is executing simple harmonic motion, and in each case find the period of the motion, the value of x when $t = 0$ and the distance from the mean position to the point O.

a $x = 2\sin 4t$, **b** $x = 5\cos 3t$,

c $x = 2\cos 2t + 4\sin 2t$, **d** $x = 1 + 3\sin 5t$.

12 Particles A and B are each executing simple harmonic motion.

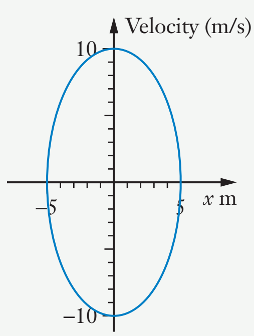
The displacement, x metres, from the respective mean positions, at time t seconds is given by

$x = c\sin k_1 t$ for particle A

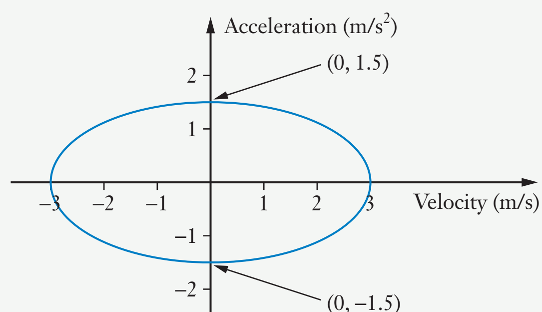
and $x = d\sin k_2 t$ for particle B, (c, k_1, d and k_2 all positive constants).

A graph for each motion is shown below.

Particle A



Particle B



Find c, d, k_1, k_2 and the time period for each particle.

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